Fault tolerant control based on set-theoretic methods

DTU guest presentation

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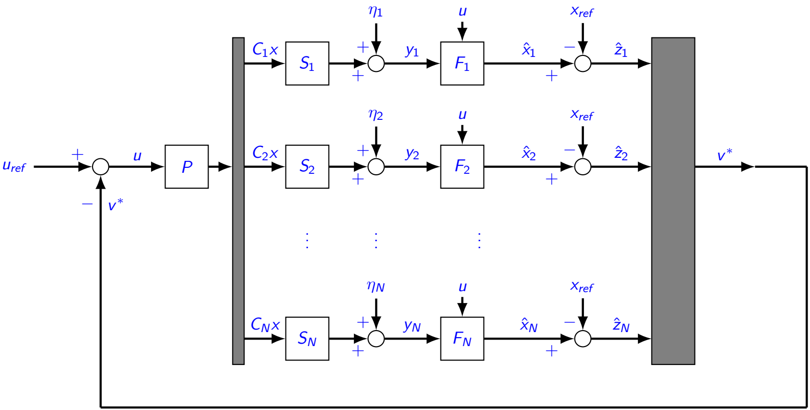
Outline

1. Fault tolerant control based on set-theoretic methods
2. Set theoretic elements
3. Mixed integer programming elements
4. Conclusions and future directions

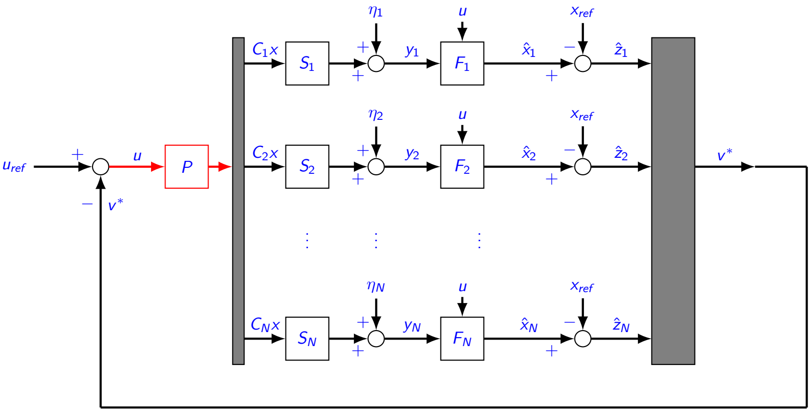
Outline

1. Fault tolerant control based on set-theoretic methods
   - Problem statement
   - FDI implementation
   - Control strategies
   - Extensions
2. Set theoretic elements
3. Mixed integer programming elements
4. Conclusions and future directions

Multisensor scheme



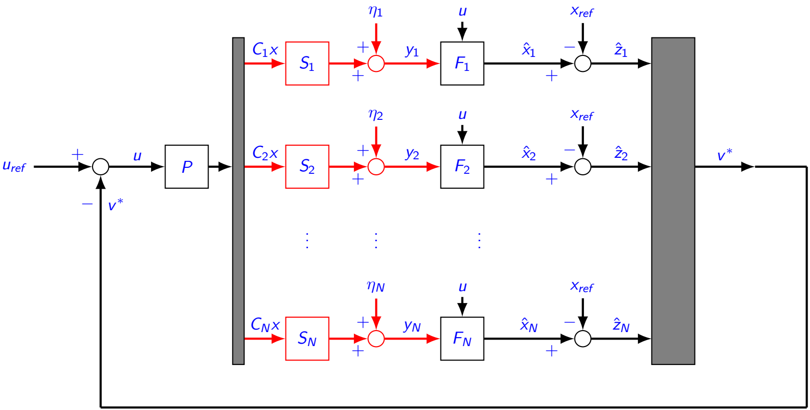
Multisensor scheme – plant



$$x^+ = Ax + Bu + Ew$$

- LTI system
- bounded noise: $w \in W$

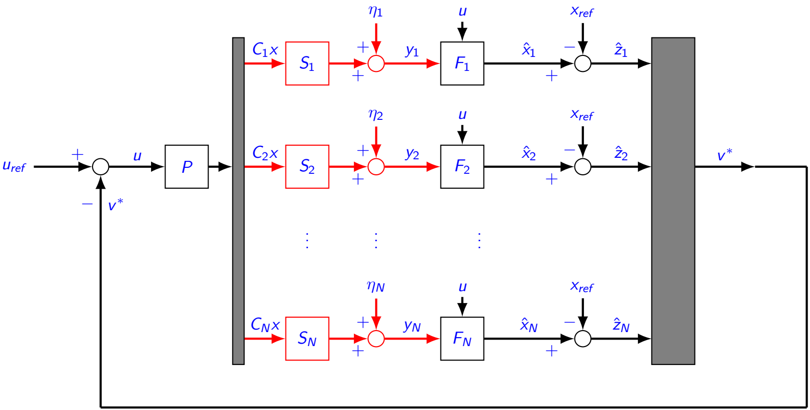
Multisensor scheme – sensors



$$y_i = C_i x + \eta_i$$

- static and redundant sensors
- bounded noise: $\eta_i \in N_i$

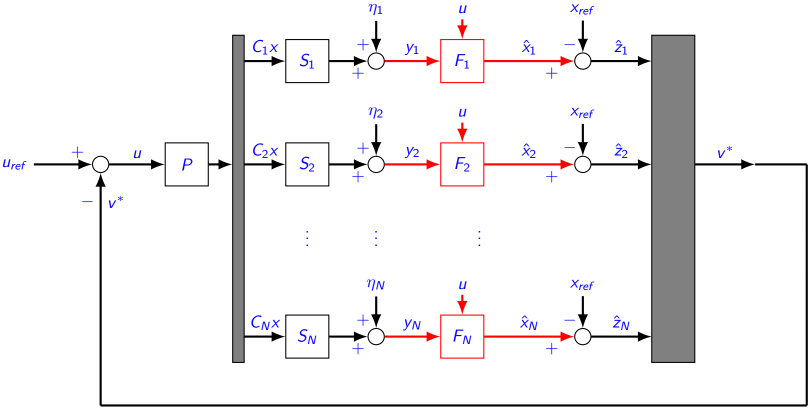
Multisensor scheme – fault scenario



$$y_i = C_i x + \eta_i \;\underset{RECOVERY}{\overset{FAULT}{\rightleftharpoons}}\; y_i = 0 \cdot x + \eta_i^F$$

- bounded noise: $\eta_i^F \in N_i^F$
- abrupt faults
- known model of the fault

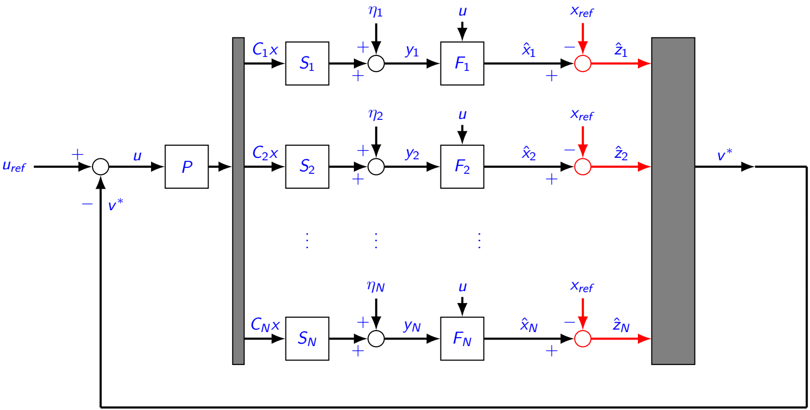
Multisensor scheme – estimates



- LTI estimators

$$\hat{x}_i^+ = A\hat{x}_i + Bu + L_i\left(y_i - C_i\hat{x}_i\right)$$

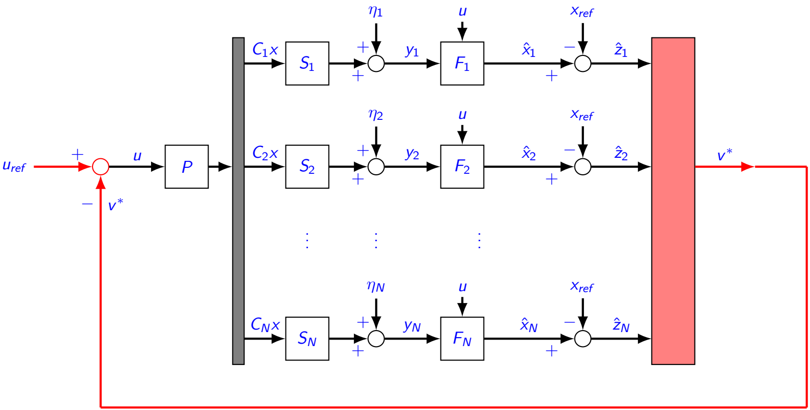
Multisensor scheme – tracking error



$$\hat{z}_i = \hat{x}_i - x_{ref}$$

- minimize tracking error

Multisensor scheme – controller



$$u = u_{ref} + v$$

- switch (and not fusion)
- fix gain + reference governor
- MPC strategies

Modeling equations

- plant dynamics

$$x^+ = Ax + Bu + Ew$$

- reference signal

$$x_{ref}^+ = Ax_{ref} + Bu_{ref}$$

- plant tracking error

$$z^+ = x - x_{ref} = Az + B\underbrace{(u - u_{ref})}_{v} + Ew$$

- estimations of the state

$$\hat{x}_i^+ = (A - L_i C_i)\,\hat{x}_i + Bu + L_i\,(y_i - C_i\hat{x}_i)$$

- estimations of the tracking error

$$\hat{z}_i = \hat{x}_i - x_{ref}$$

Set separation conditions

Reminder:

- $z = x - x_{ref}$
- $y_i = C_i x + \eta_i \underset{RECOVERY}{\overset{FAULT}{\rightleftharpoons}} y_i = 0 \cdot x + \eta_i^F$
- $\eta_i \in N_i$, $\eta_i^F \in N_i^F$

Consider the residual signal

$$r_i = y_i - C_i x_{ref}, \qquad \begin{cases} r_i^H = C_i z + \eta_i \\ r_i^F = -C_i x_{ref} + \eta_i^F \end{cases}$$

Set separation condition:

$$\left(\{C_i z\} \oplus N_i\right) \cap \left(\{-C_i x_{ref}\} \oplus N_i^F\right) = \emptyset$$

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Set separation condition:

$$\left(C_i\ S_z\ \oplus N_i\right) \cap \left(-C_i\ X_{ref}\ \oplus N_i^F\right) = \emptyset$$

Assume that:

- $z \in S_z$
- $x_{ref} \in X_{ref}$

Set separation conditions

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Assume that:

- $z \in\ S_z$
- $x_{ref} \in\ X_{ref}$

$$R_i^H \cap R_i^F = \emptyset \longrightarrow \begin{cases} r_i \in R_i^H \leftrightarrow y_i = C_i x + \eta_i \\ r_i \in R_i^F \leftrightarrow y_i = 0 \cdot x + \eta_i^F \end{cases}$$

Auxiliary sets

- boundedness assumptions: N_i, N_i^F, W
- X_{ref} – set for the reference signal
- $\tilde{S}_i$ – invariant set for the state estimation error
- S_z – invariant set for the plant tracking error

State estimation error:

$$\tilde{x}_i^+ = x^+ - \hat{x}_i^+ = (A - L_i C_i)\,\tilde{x}_i + \begin{bmatrix} E & -L_i \end{bmatrix} \begin{bmatrix} w \\ \eta_i \end{bmatrix}$$

Plant tracking error (for fix gain $v = -K\hat{z}_l$):

$$z^+ = (A - BK)\,z + \begin{bmatrix} E & BK \end{bmatrix} \begin{bmatrix} w \\ \tilde{x}_l \end{bmatrix}$$

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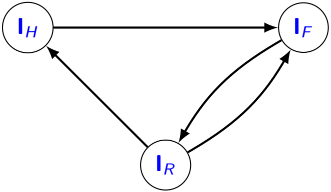
Sensor partitioning

- $\mathbf{I}_H = \{i \in \mathbf{I}_H^- : r_i \in R_i^H\} \cup \{i \in \mathbf{I}_R^- : S_i^R \subseteq \tilde{S}_i,\ r_i \in R_i^H\}$
- $\mathbf{I}_F = \{i \in \mathcal{I} : r_i \notin R_i^H\}$
- $\mathbf{I}_R = \mathcal{I} \setminus (\mathbf{I}_H \cup \mathbf{I}_F)$.

$$\mathcal{I} = \mathbf{I}_H \cup \mathbf{I}_F \cup \mathbf{I}_R$$



	$\mathbf{I}_H$	$\mathbf{I}_F$	$\mathbf{I}_R$
$\tilde{x}_i \in \tilde{S}_i$	✓	—	X
$r_i \in R_i^H$	✓	X	✓

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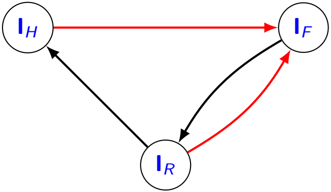

$$r_i \in R_i^H \longrightarrow r_i \notin R_i^H$$

	$\mathbf{I}_H$	$\mathbf{I}_F$	$\mathbf{I}_R$
$\tilde{x}_i \in \tilde{S}_i$	✓	—	X
$r_i \in R_i^H$	✓ $\longrightarrow$	X	$\longleftarrow$ ✓

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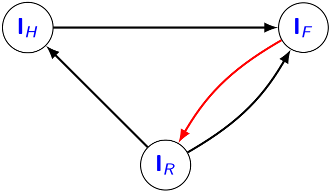

$$r_i \notin R_i^H \longrightarrow r_i \in R_i^H$$

	$\mathbf{I}_H$	$\mathbf{I}_F$	$\mathbf{I}_R$
$\tilde{x}_i \in \tilde{S}_i$	✓	—	X
$r_i \in R_i^H$	✓ ⟵	X	✓

Sensor partitioning

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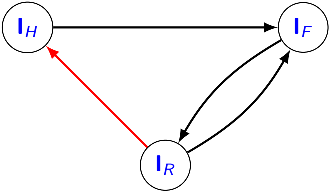


$$\tilde{x}_i \notin \tilde{S}_i \longrightarrow \tilde{x}_i \in \tilde{S}_i$$

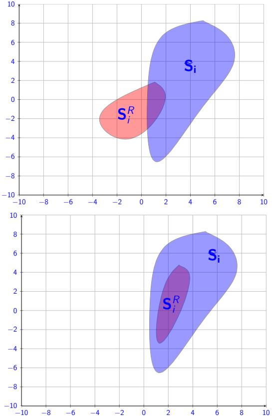
	$\mathbf{I}_H$	$\mathbf{I}_F$	$\mathbf{I}_R$
$\tilde{x}_i \in \tilde{S}_i$	✓	—	X
$r_i \in R_i^H$	✓	X	✓

Recovery – preliminaries

Conditions for recovery acknowledgment ($\mathbf{I}_R \rightarrow \mathbf{I}_H$)

- $r_i \in R_i^H$ – residual
- $\tilde{x}_i \in \tilde{S}_i$ – estimation error

$\tilde{x}_i = x - \hat{x}_i$ is not measurable but we construct S_i^R such that $\tilde{x}_i \in S_i^R$



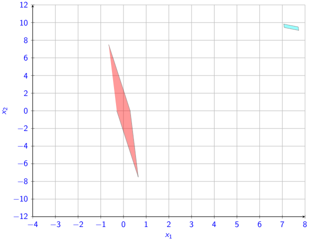
Strategies:

- necessary conditions
- sufficient conditions

- $\tilde{x}_i \in S_i^R$, a necessary condition for $\tilde{x}_i \in \tilde{S}_i$ is $S_i^R \cap \tilde{S}_i \neq \emptyset$
- $\tilde{x}_i \in S_i^R$, a sufficient condition for $\tilde{x}_i \in \tilde{S}_i$ is $S_i^R \subseteq \tilde{S}_i$

Recovery – validation

$$\mathbf{I}_R \xrightarrow{i} \mathbf{I}_H : \quad (i \in \mathbf{I}_R^-) \wedge (S_i^R \subseteq \tilde{S}_i) \wedge (r_i \in R_i^H)$$



Issues:

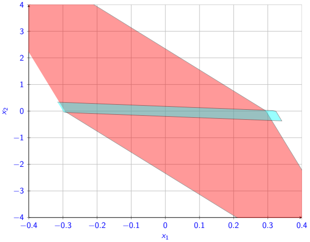
- gap time
- inclusion validation

Strategies (during faulty functioning):

- gap time
  - keep the original dynamics of the estimator (Olaru et al. [2009])
  - change the dynamics of the estimator (Stoican et al. [2010b])
  - reset the estimation ($\hat{x}_i^\circ = x_{ref}$ or $\hat{x}_i^\circ = \hat{x}_l$)
- inclusion validation
  - wait for the validation of the inclusion
  - compute the reachable set of S_i^R and observe when the inclusion is validated

Recovery – validation

$$\mathbf{I}_R \xrightarrow{i} \mathbf{I}_H : \quad (i \in \mathbf{I}_R^-) \wedge (S_i^R \subseteq \tilde{S}_i) \wedge (r_i \in R_i^H)$$



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- inclusion validation
  - wait for the validation of the inclusion
  - compute the reachable set of S_i^R and observe when the inclusion is validated

Illustrative example

Consider the interdistance example with dynamics

$$x^+ = \underbrace{\begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0.5 \end{bmatrix}}_{B} u + \underbrace{\begin{bmatrix} 0 \\ 0.1 \end{bmatrix}}_{E} w$$

with $W = \{w : \; |w| \leq 0.2\}$.

$$C_1 = \begin{bmatrix} 0.35 & 0.25 \end{bmatrix}, \; |\eta_1| \leq 0.15, \; |\eta_1^F| \leq 1$$
$$C_2 = \begin{bmatrix} 0.30 & 0.80 \end{bmatrix}, \; |\eta_2| \leq 0.1, \; |\eta_2^F| \leq 1$$
$$C_3 = \begin{bmatrix} 0.35 & 0.25 \end{bmatrix}, \; |\eta_3| \leq 0.1, \; |\eta_3^F| \leq 0.3.$$

Illustrative example – FDI validation

Consider the interdistance example with dynamics

$$x^+ = \underbrace{\begin{bmatrix}1 & 0.1\\ 0 & 1\end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix}0\\ 0.5\end{bmatrix}}_{B} u + \underbrace{\begin{bmatrix}0\\ 0.1\end{bmatrix}}_{E} w$$

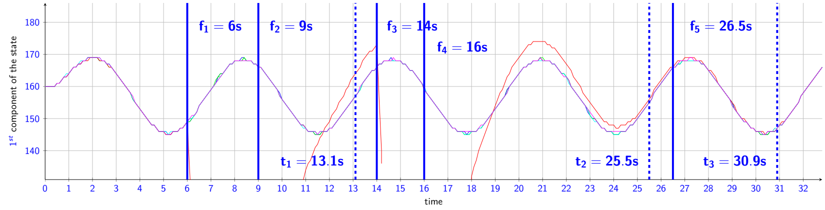
with $W = \{w : |w| \le 0.2\}$.

$$R_1^H = \{r_1 : -22.9 \le r_1 \le 22.9\},$$
$$R_2^H = \{r_2 : -19.8 \le r_1 \le 19.8\},$$
$$R_3^H = \{r_3 : -22.9 \le r_1 \le 22.9\}.$$

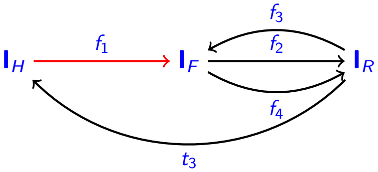
$$R_1^F = \{r_1 : -58.9 \le r_1 \le -49.8\},$$
$$R_2^F = \{r_2 : -53.9 \le r_1 \le -39.2\},$$
$$R_3^F = \{r_3 : -58.1 \le r_1 \le -50.5\}.$$

Illustrative example – recovery validation

Sensors estimations for test case when 3^{th} sensor fails twice at f_1 and f_3 respectively:



set transitions for sensor with index 3



Control strategies

Issues related to FTC:

- faults may impose control law reconstruction
- under fault, the objectives may need to change

Control strategies:

- fix gain with reference governor
- MPC formulation

Both strategies use the separation condition

$$\left(\{C_i z\} \oplus N_i\right) \cap \left(\{-C_i x_{ref}\} \oplus N_i^F\right) = \emptyset$$

to assure exact FDI.

FDI adjusted reference governor

Fix z and let x_{ref} be the decision variable:

$$D_{x_{ref}} \triangleq \left\{x_{ref} : \left(\{-C_i x_{ref}\} \oplus N_i^F\right) \cap \left(C_i S_z \oplus N_i\right) = \emptyset,\ i = 1 \dots N\right\}.$$

Reference governor (Stoican et al. [2010d]):

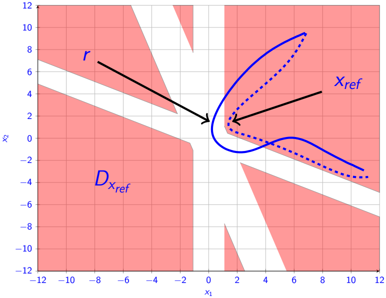
$$u^*_{ref[0,\tau-1]} = \underset{u_{ref[0,\tau-1]}}{\arg\min} \sum_{i=0}^{\tau-1} \left(||r_{[i]} - x_{ref[i]}||_{Q_r} + ||u_{ref[i]}||_{R_r} \right)$$

subject to:

$$x^+_{ref[i]} = A x_{ref[i]} + B u_{ref[i]}$$
$$x^+_{ref[i]} \in D_{x_{ref}}$$

Characteristics:

- fix gain
- flexible reference



MPC with FDI feasibility guarantees

Fix x_{ref} and let z be the decision variable:

$$D_z \triangleq \left\{z : \left(\{C_i z\} \oplus N_i\right) \cap \left(\{-C_i X_{ref}\} \oplus N_i^F\right) = \emptyset,\ i = 1 \ldots N\right\}$$

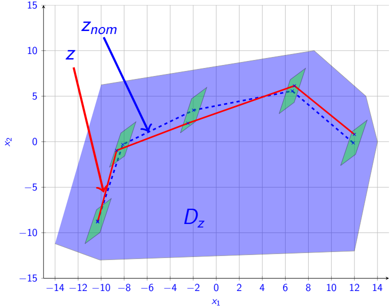
into the MPC formulation:

$$v^*_{[0,\tau-1]} = \underset{v_{[0,\tau-1]}}{\arg\min} \left\{ \sum_{i=0}^{\tau-1} \left(||z_{[i]}||_Q + ||v_{[i]}||_R\right) + ||z_{[\tau]}||_P \right\}$$

subject to:

$$z^+_{[i]} = Az_{[i]} + Bv_{[i]} + E\, w_{[i]}$$

$$z^+_{[i]} \in D_z$$



Issues:

- stability guarantees
- numerical complexity (reachable sets)

MPC with FDI feasibility guarantees

Fix x_{ref} and let z be the decision variable:

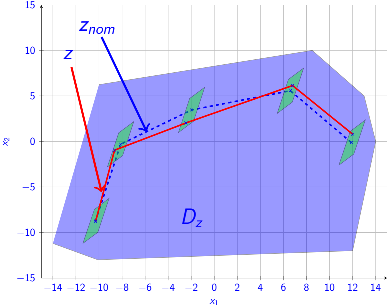
$$D_z \triangleq \left\{ z : \left(\{C_i z\} \oplus N_i \right) \cap \left(\{-C_i X_{ref}\} \oplus N_i^F \right) = \emptyset,\ i = 1 \dots N \right\}$$

into the tube-MPC formulation($z \in \{z_{nom}\} \oplus S_z$):

$$v^*_{nom[0,\tau-1]} = \underset{v_{nom[0,\tau-1]}}{\arg\min} \left\{ \sum_{i=0}^{\tau-1} \left(||z_{nom[i]}||_Q + ||v_{nom[i]}||_R \right) + ||z_{nom[\tau]}||_P \right\}$$

subject to:

$$z^+_{nom[i]} = A z_{nom[i]} + B v_{nom[i]}$$
$$z^+_{nom[i]} \in D_z \ominus S_z$$



Issues:

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MPC with FDI feasibility guarantees (II)

Reminder:

- tracking error estimation: $\hat{z}_i = \hat{x}_i - x_{ref}$
- tracking error: $z = x - x_{ref}$
- estimation error: $\tilde{x}_i = x - \hat{x}_i$

$z_{nom} = ?$

which leads to:

$$z = \hat{z}_i + \tilde{x}_i$$

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which leads to:

$$z \in \{\hat{z}_i\} \oplus \tilde{S}_i$$

Several directions ($i \in \mathbf{I}_H$ and $j = 0 \dots \tau - 1$):

- individual merit
  keep the same sensor during the prediction horizon: $z_{nom[j]} = \hat{z}_{i[j]}$
- relay race
  check the sensor index at each iteration: $z_{nom[j]} = \hat{z}_{i_j[j]}$
- collaborative scenario
  consider a convex sum of the sensors (at least in the terminal step):
  $z_{nom[\tau]} = conv\{\hat{z}_{i[\tau]}\}$

Can be applied also for fix control laws $v = -K\hat{z}_i$

The estimation error as residual signal

Consider the residual signal as

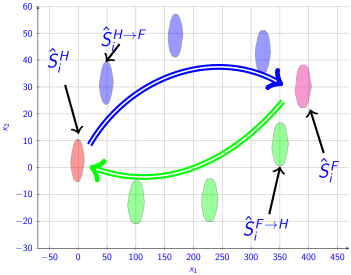
$$r_i = \hat{z}_i$$

The residual sets for healthy to faulty transitions are:

- $R_i^H = \hat{S}_i^H$ (the invariant set of dynamics $\hat{z}_i$ under healthy functioning)
- $R_i^F = \hat{S}_i^{H\to F}$ (the one-step reachable set of $\hat{S}_i^H$ under faulty functioning for $\hat{z}_i$)

Particularities:

- requires persistent faults
- recovers the entire information
- permits passive FTC
- has filter behavior



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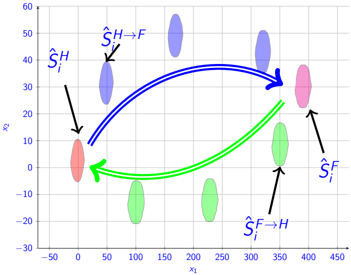
$$r_i = \hat{z}_i$$

The residual sets for faulty to healthy transitions are:

- $R_i^H = \hat{S}_i^F$ (the invariant set of dynamics $\hat{z}_i$ under faulty functioning)
- $R_i^F = \hat{S}_i^{F\to H}$ (the one-step reachable set of $\hat{S}_i^F$ under healthy functioning for $\hat{z}_i$)

Particularities:

- requires persistent faults
- recovers the entire information
- permits passive FTC
- has filter behavior

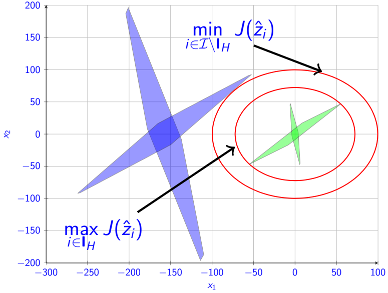


Passive FTC implementation

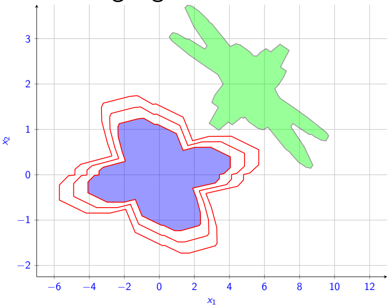
For a cost function $J(\cdot)$ passive FTC is possible if:

$$\max_{i \in \mathbf{I}_H} J(\hat{z}_i) < \min_{i \in \mathcal{I} \setminus \mathbf{I}_H} J(\hat{z}_i)$$

quadratic function



gauge function



$$J(\hat{z}_i) = \hat{z}_i^T P \hat{z}_i$$

$$J(\hat{z}_i) = J^* \{\lceil \rho_H(\hat{z}_i) \rceil - 1\} + J^* \lceil \rho_H(\hat{z}_l) \rceil$$

Passive FTC implementation

For a cost function $J(\cdot)$ passive FTC is possible if:

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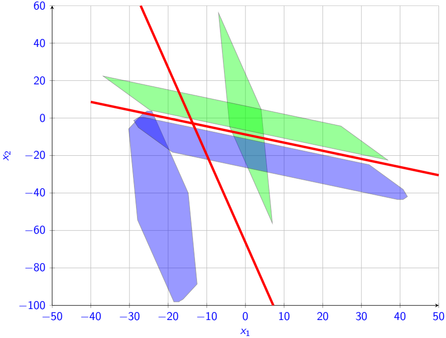

Not always possible!

Extended residual

Consider a receding observation horizon of length τ with extended residual

$$r_i = y_{i[-\tau,0]} - C_{i,\tau} x_{ref[-\tau,0]} - \Gamma_{i,\tau} v_{[-\tau,0]}$$

which leads to:

$$\begin{aligned} r_i^H &= \quad \Theta_{i,\tau} z_{[-\tau]} + \Phi_{i,\tau} w_{[-\tau,0]} + \eta_{i[-\tau,0]} \\ r_i^F &= -\Theta_{i,\tau} x_{ref[-\tau]} - \Gamma_{i,\tau} \left(u_{ref[-\tau,0]} + v_{[-\tau,0]} \right) + \eta^F_{i[-\tau,0]} \end{aligned}$$

Set separation guarantee for FDI:

$$-\Theta_{i,\tau} \left(z + x_{ref[-\tau]} \right) - \Gamma_{i,\tau} \left(u_{ref[-\tau,0]} + v_{[-\tau,0]} \right) \notin P_i$$

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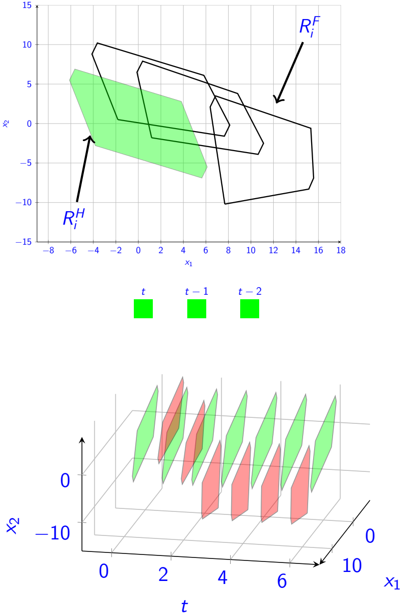
All control parameters influence the capacity of fault detection

Extended residual (II)

Particularities:

- requires persistent faults (only for τ instants)
- recovers the entire information
- enhances the separation conditions
- adds delay in the control design
    - stability harder to enforce
    - maximizes FDI admissible space



Influences of extended residuals in RC design

General condition for FDI validation:

$$\mathbb{D}_{ref} \triangleq \left\{ -\Theta_{i,\tau} \left(z + x_{ref[-\tau]}\right) - \Gamma_{i,\tau} \left(u_{ref[-\tau,0]} + v_{[-\tau,0]} \right) \notin P_i \right\}$$

Control strategies:

- fix gain with delayed information ($v_{[-\tau,0]} = -K\hat{z}_{i[-2\tau,-\tau]}$) leads to condition:

$$-\Theta_{i,\tau} x_{ref[-\tau]} - \Gamma_{i,\tau} u_{ref[-\tau,0]} \notin P_i \ominus \left\{-KS_{z_{[-2\tau,-\tau]}}\right\} \ominus S_z$$

  to be used in a reference governor.
- MPC formulation:

$$(u^*_{ref}, v^*) = \underset{u_{ref[0,\sigma]}, v_{[0,\sigma]}}{\arg\min} \sum_{j=0}^{\sigma} f\left(x_{ref[j]}, z_{[j]}, u_{ref[j]}, v_{[j]}\right)$$

  subject to:

$$\begin{aligned} x^+_{ref[j]} &= Ax_{ref[j]} + Bu_{ref[j]} \\ z^+_{[j]} &= Az_{[j]} + Bv_{[j]} + Ew_{[j]} \\ \left(x_{ref[j-\tau]}, u_{ref[j-\tau,j]}, v_{[j-\tau,j]}, z_{[j]}\right) &\in \mathbb{D}_{ref[j]} \end{aligned}$$

FDI adjustment for fix gain control

Control strategy for fix gain feedback:

- instead of computing the set invariant for a given dynamics we try to determine the dynamics that make a given set invariant
- for a bounded reference $x_{ref} \in X_{ref}$ the feasible tracking error region is given by

$$D_z \triangleq \left\{ z : \left(\{C_i z\} \oplus N_i \right) \cap \left(\{-C_i X_{ref}\} \oplus N_i^F \right) = \emptyset,\ i = 1 \dots N \right\}$$

Take $S_z \subseteq D_z$ and enforce its invariance as a parameter after K (Stoican et al. [2010a]):

$$S_z \ = \{z :\ Hz \leq K\} \subseteq D_z$$

$$z^+ = (A - B\ K\ )z + \begin{bmatrix} E & B\ K \end{bmatrix} \begin{bmatrix} w \\ \tilde{x}_l \end{bmatrix}$$

$$\epsilon^* = \max_{l} \min_{\substack{K,H,\epsilon \\ \epsilon \geq 0 \\ HF_z = F_z(A-BK) \\ H\theta_z + F_z B_{z,l}\delta_{z,l} \leq \epsilon\theta_z \\ \delta_{z,l} \in \Delta_{z,l}}} \epsilon$$

if $\epsilon^* \leq 1$ the solution is feasible

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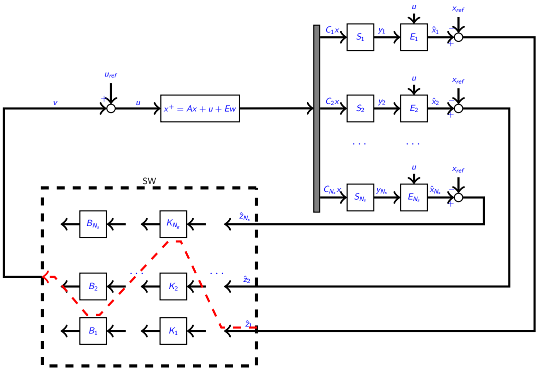
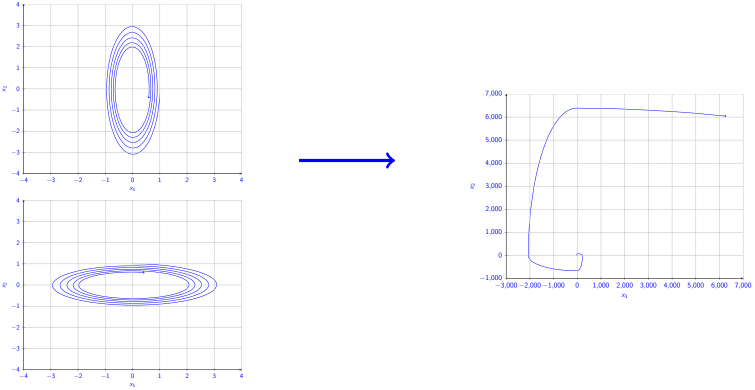
From multisensor to multiple loops



- the same principles hold for actuator/subsystems faults
- issues to be considered:
  - computations more difficult (star-shaped sets)
  - the system becomes switched

Switched systems particularities

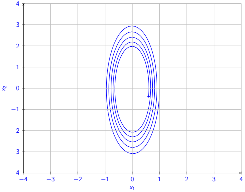
Note (Branicky [1994]): A switched system may not be stable even if all its subsystems are stable:

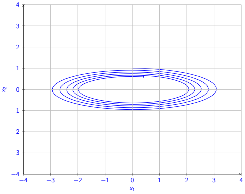


Switched systems particularities

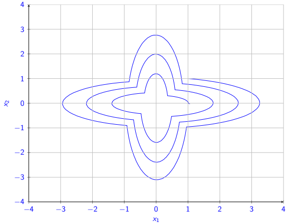
Theorem (Geromel and Colaneri [2006])

Let there be the switched system $x^+ = A_i x$ and assume that:





$$\begin{cases} P_i > 0 \\ A_i' P_i A_i + P_i \leq 0 \\ {A_i'}^T P_j {A_i}^T < P_i \quad \forall j \neq i \end{cases}$$



then the system is globally stable for any switch occurring at moments greater or equal with T.

Difficulty: RPI construction for switched systems (Stoican et al. [2010c])

Outline

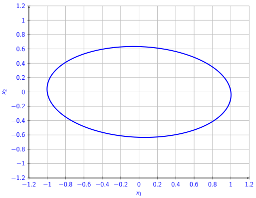
Families of sets – generalities

Various families of sets in control:

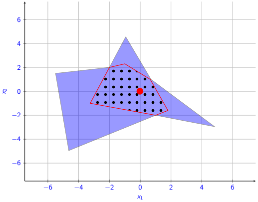
- ellipsoids (Kurzhanskiĭ and Vályi [1997])
- polytopes/zonotopes (Motzkin et al. [1959])
- (B)LMIs (Nesterov and Nemirovsky [1994])
- star-shaped sets (Rubinov and Yagubov [1986])

Issues to be considered:

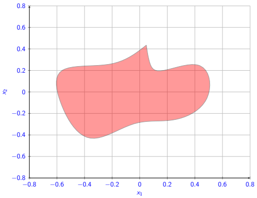
- flexibility of the representation
- numerical implementation



$x^T Qx \leq \gamma$



$Kern(S) \neq \emptyset$



$G(x) \leq 0$

Families of sets – generalities

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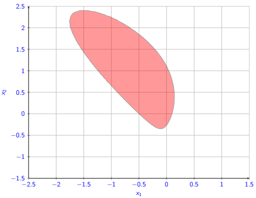
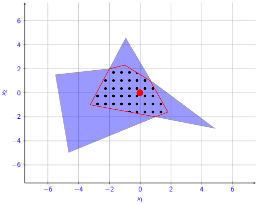
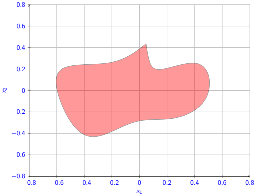






$A_0 + \sum x_i A_i \succ 0$ $\quad$ $Kern(S) \neq \emptyset$ $\quad$ $G(x) \leq 0$

Families of sets – polyhedral/zonotopic sets (more "structured")

Best compromise: polytopic(zonotopic) sets

Polyhedral sets:

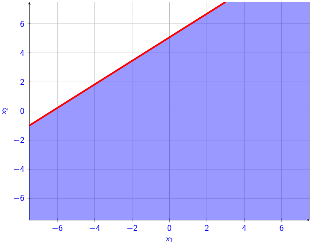
- dual representation
  - half-space:

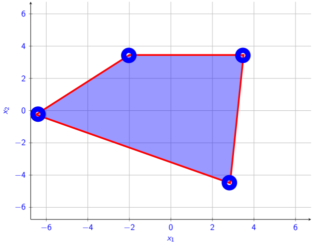
$$h_i x \leq k_i,\ i = 1 \ldots N_h$$

  - vertex:

$$\sum_i \alpha_i v_i, \alpha_i \geq 0, \sum_i \alpha_i = 1, i = 1 \ldots N_v$$

- efficient algorithms for set containment problems (Gritzmann and Klee [1994])
- can approximate any convex shape (Bronstein [2008])





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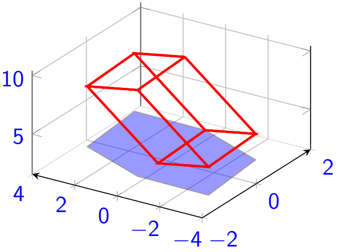
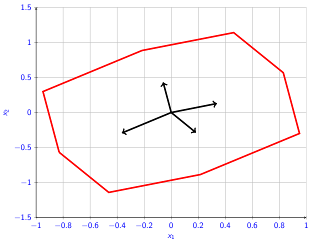


Zonotopic sets:

- obtained as
  - hypercube projection
  - Minkowski sum of generators
- additional representation
  - generator form:

$$\sum_i \lambda_i g_i,\ |\lambda_i| \leq 1,\ i = 1 \ldots N_g$$

  - compact representation
- limited to symmetric objects



Invariance notions

Consider a system in $\mathbb{R}^n$

$$x^+ = f(x, \delta)$$

with disturbances bounded by the set $\Delta \subset \mathbb{R}^n$.

Definition (RPI set)

A set Ω is called robust positive invariant (RPI) iff

$$f(\Omega, \Delta) \subseteq \Omega.$$

The minimal RPI set (which is contained in all the RPI sets) can be defined as:

$$\Omega_\infty = \underbrace{f(f(\ldots, \Delta), \Delta)}_{\infty \text{ iterations}} = \lim_{k \to \infty} f^{(k)}(0, \Delta).$$

Invariance notions

Consider a LTI system in $\mathbb{R}^n$

$$x^+ = Ax + B\delta$$

with A a Schur matrix and disturbances bounded by the set $\Delta \subset \mathbb{R}^n$.

Definition (RPI set)

A set Ω is called robust positive invariant (RPI) iff

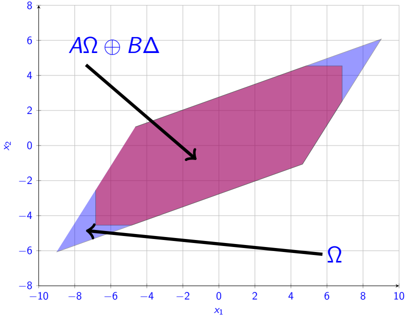
$$A\Omega \oplus B\Delta \subseteq \Omega.$$

The minimal RPI set (which is contained in all the RPI sets) can be defined as:

$$\Omega_\infty = \bigoplus_{i=0}^{\infty} A^i B\Delta.$$

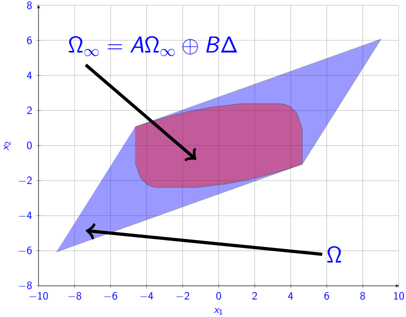
Invariance notions – exemplification

RPI set



$$A\Omega \oplus B\Delta \subseteq \Omega$$

mRPI set



$$\Omega_\infty = \bigoplus_{i=0}^{\infty} A^i B\Delta$$

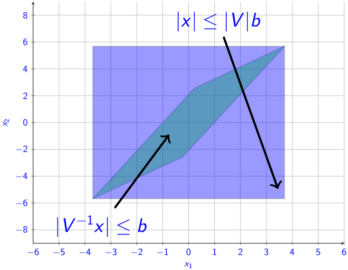
Ultimate bounds for zonotopic sets

Theorem (Ultimate bounds – Kofman et al. [2007])

For sytem $x^+ = Ax + B\delta$ with the Jordan decomposition $A = V\Lambda V^{-1}$ and assuming that $|\delta| \leq \bar{\delta}$ we have that the set $\Omega_{UB}(\epsilon)$ is RPI.

Particularities:

- explicit linear formulations
- "good" approximation of the mRPI set
- can be extended to various degenerate cases (Haimovich et al. [2008], Kofman et al. [2008])



$$\Omega_{UB}(\epsilon) = \left\{x : |V^{-1}x| \leq (I - |\Lambda|)^{-1}|V^{-1}B|\bar{\delta} + \epsilon\right\}$$

Ultimate bounds for zonotopic sets

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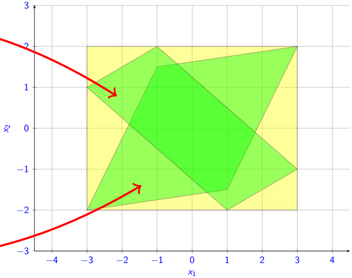
$$\delta_1 \in \Delta_1 \, , \, |\delta_1| \leq \bar{\delta}$$

$$\Rightarrow$$

$$\delta_2 \in \Delta_2 \, , \, |\delta_2| \leq \bar{\delta}$$



Sets with the same bounding box will give the same UBI set for a given dynamic.

Improvement (Stoican et al. [2011a]): use zonotopic sets for describing the disturbance.

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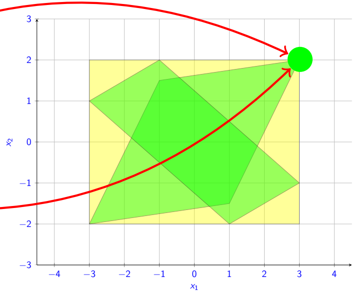

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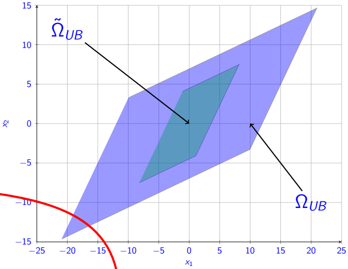
For a zonotopic perturbation

$$\Delta = C\,\mathbb{B}^m_\infty$$

the dynamics become

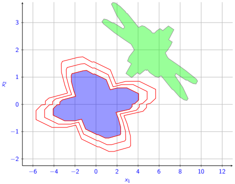
$$x^+ = Ax + B\delta = Ax + BCw$$



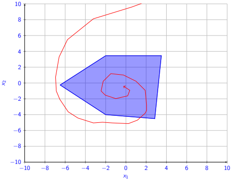
and the UBI set becomes:

$$\tilde{\Omega}_{UB}(\epsilon) = \left\{x : |V^{-1}x| \le (I - |\Lambda|)^{-1}|V^{-1}B\ C\ |\mathbf{1} + \epsilon\right\}$$

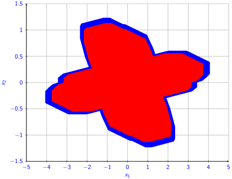
Other set theoretic topics



- set separation between sets
  - through a separating hyperplane
  - through a barrier function



- upper bound for the inclusion time
  - particular bounds for a given attractive set



- RPI description for particular dynamics
  - switched/with delay
  - cyclic invariance

Outline

MIP – Preliminaries

Set separation problems usually lead to nonconvex feasible regions for optimization problems (usually, the complement of a polyhedral set):

$$x^* = \arg\min_{x \notin P} J(x)$$

where

$$P = \{x : \ h_i x \leq k_i, \ i = 1 \ldots N\}.$$

The goal is to reduce the number of binary variables in the extended representation.

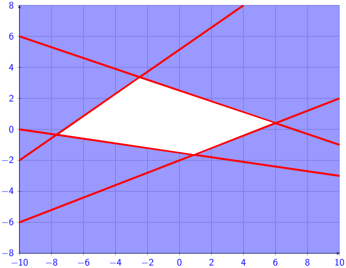
MIP – Basic idea

Linear extended representation:

$$-h_i x \leq -k_i + M\alpha_i, \quad i = 1: N$$

$$\sum_{i=1}^{i=N} \alpha_i \leq N - 1$$

with $(\alpha_1, \ldots, \alpha_N) \in \{0, 1\}^N$



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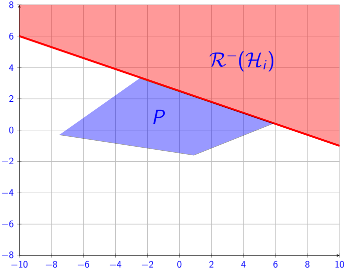


Any of the regions $\mathcal{R}^-(\mathcal{H}_i)$ of $\mathcal{C}(P)$ can be obtained by a suitable choice of binary variables

$$\mathcal{R}^-(\mathcal{H}_i) \longleftrightarrow (\alpha_1, \ldots, \alpha_N)^i \triangleq (1, \ldots, 1, \underbrace{0}_{i}, 1, \ldots, 1)$$

MIP – Basic idea

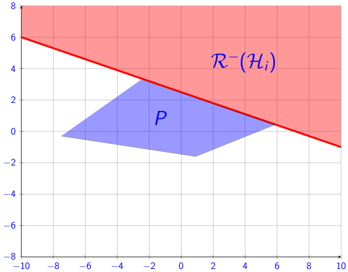
Linear extended representation:

$$-h_i x \leq -k_i + M\alpha_i(\lambda), \quad i = 1 : N$$

$$0 \leq \beta_l(\lambda)$$

with $\alpha_i(\lambda) : \{0,1\}^{N_0} \rightarrow \{0\} \cup [1, \infty)$
and

$$N_0 = \lceil log_2 N \rceil$$



Any of the regions $\mathcal{R}^{-}(\mathcal{H}_i)$ of $\mathcal{C}(P)$ can be obtained by a suitable choice of binary variables (Stoican et al. [2011b])

$$\mathcal{R}^{-}(\mathcal{H}_i) \longleftrightarrow (\lambda_1, \ldots, \lambda_{N_0})^i$$

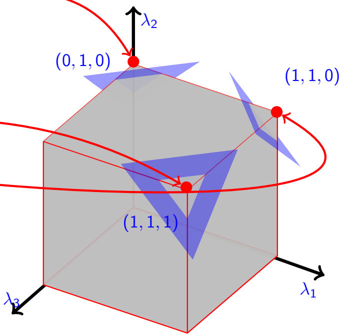
MIP – Basic idea

Linear extended representation:

$$-h_i x \leq -k_i + M\alpha_i(\lambda), \quad i = 1 : N$$
$$0 \leq \beta_l(\lambda)$$

with $\alpha_i(\lambda) : \{0,1\}^{N_0} \to \{0\} \cup [1, \infty)$
and

$$N_0 = \lceil log_2 N \rceil$$



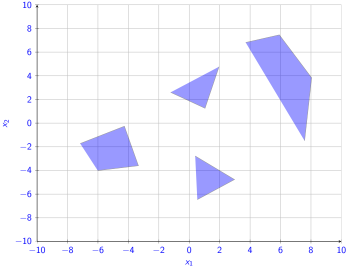
For any $\lambda \in \{0,1\}^{N_0}$ unallocated to a region $\mathcal{R}^-(\mathcal{H}_i)$, the MI representation degenerates to the entire space $\mathbb{R}^n$.

Solution: add constraints that make the unallocated tuples infeasible

MIP – Non-connected regions

Consider the complement $\mathcal{C}(\mathbb{P}) = cl(\mathbb{R}^n \setminus \mathbb{P})$ of a union of polyhedral sets $\mathbb{P} = \bigcup_j P_j$.

$$\mathcal{A}(\mathbb{H}) = \bigcup_{l=1,\dots,\gamma(N)} \underbrace{\left(\bigcap_{i=1}^{N} R^{\sigma_l(i)}(\mathcal{H}_i)\right)}_{A_l}$$

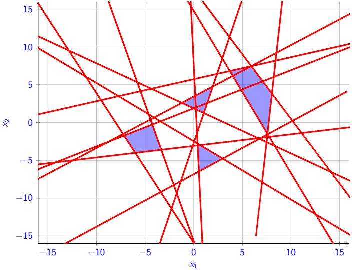


Using the hyperplanes $\mathcal{H}_i$ we partition the space into disjoint cells A_l.

MIP – Non-connected regions

Consider the complement $\mathcal{C}(\mathbb{P}) = cl(\mathbb{R}^n \setminus \mathbb{P})$ of a union of polyhedral sets $\mathbb{P} = \bigcup_j P_j$.

$$A_l \begin{cases} & \dots \\ \sigma_l(1)h_1x & \leq \sigma_l(1)k_1 + M\alpha_l(\lambda) \\ & \vdots \\ \sigma_l(N)h_Nx & \leq \sigma_l(N)k_N + M\alpha_l(\lambda) \\ & \dots \\ & 0 \leq \beta_l(\lambda) \end{cases}$$



Using the same procedure we associate a linear combination of binary variables $\alpha_l(\lambda)$ to each cell (Stoican et al. [2011c]).

MIP – Non-connected regions

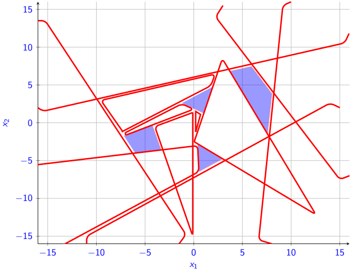
Consider the complement $\mathcal{C}(\mathbb{P}) = cl(\mathbb{R}^n \setminus \mathbb{P})$ of a union of polyhedral sets $\mathbb{P} = \bigcup_j P_j$.

$$\dots$$

$$A_l \begin{cases} \sigma_l(1)h_1 x & \leq \sigma_l(1)k_1 + M\alpha_l(\lambda) \\ & \vdots \\ \sigma_l(N)h_N x & \leq \sigma_l(N)k_N + M\alpha_l(\lambda) \end{cases}$$

$$\dots$$

$$0 \leq \beta_l(\lambda)$$



The number of cells can be reduced through merging procedures.

Outline

Conclusions

- invariant sets offer a robust FTC approach
- a countable number of sensor fault scenarios can be arbitrary chosen
- a global view in considering the effects of the FDI mechanism
- extensions to MPC
- good balance between computational effort and precision
- robust fault detection

Future directions – set theoretic elements

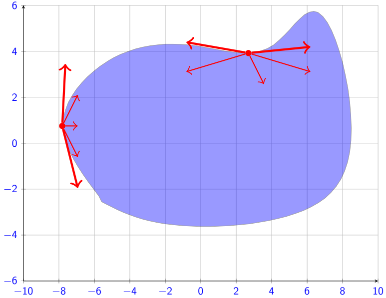
A) "Wish list" for set theoretic constructions:
- invariance computations
  - explicit formula for the boundary of the mRPI set
  - RPI sets for switched/with delay systems
- optimization problem which returns an RPI set for given dynamics and constraints (+ fix structure)
- faster algorithms for set operations (treat degenerate polyhedral cases)
- comprehensive framework for zonotopes

B) A broader perspective:
- bridge the gap with stochastic FTC by the use of probabilistic invariance
- remain under boundedness assumptions and reformulate FTC in the viability theory framework

Future directions – viability theory

Viability theory generalizes a series of geometrical notions:

- set valued maps and difference inclusions
- continuity/derivability
- set shape
- positive (control) invariance (viability/invariance kernels)



$$\begin{cases} x'(t) \in F(x(t), u(t), f(t)) \\ u(t) \in U(x(t)) \end{cases}$$

$$R_k(x) = \{u(x) \in U(x),\ F(x, u(x), f(t)) \subseteq T_K(x)\}$$

Issues:

- numerical algorithms hard to apply
- dense mathematical framework

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Thank you!

Questions ?